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ON STABILITY OF PERIODIC MOTIONS IN A PARTICULAR CRITICAL CASE*

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Liapunov's stability problem of the zero solution for a nonautonomous system of differential equations with periodic coefficients is considered in the case when the characteristic equation of the linearized system has r roots equal unity and $2q$ complex conjugate roots in modulus equal unity. Several integral relations (internal resonances) may exist between the characteristic indices and the unperturbed motion frequency. Equations are presented in standard form. Normalization is carried out by the Deprit—Hori method with a modification similar to that of Mersman /1/. Sufficient conditions of instability and asumptotic stability are obtained in the case of a system with a single internal resonance and $r=1$ and $q=1$.

Let us consider the stability problem of the zero solution of a system of nonautonomous differential equations with holomorphic right-hand sides as follows

$$dx/dt = A(t)x + \sum_{l\geqslant 2} X^{(l)}(x,t), \quad X^{(l)}(0,t)\equiv 0, \quad A(t+\omega)=A(t), \quad X^{(l)}(x,t+\omega)=X^{(l)}(x,t), \quad x\in R^{r+2q} \tag{0.1}$$

where $X^{(l)}(x,t)$ are holomorphic vector functions that are periodic in t of real period ω, and represent l-th order forms.

Let us assume that the characteristic equation of system (0.1) has r roots equal unity and $2q$ roots in modulus equal unity $(\rho_s = \exp(\pm\sqrt{-1}\omega_s),\ s=1,\ldots,q)$ to which correspond simple elementary divisors.

As shown in /2/, the stability problem of type (0.1) systems reduces to the stability analysis of the zero solution of the system of autonomous equations with $(r+q)$ zero roots to which correspond simple elementary divisors, provided that among the characteristic in indices $\lambda_s = \pm\sqrt{-1}\omega_s,\ \omega_s>0,\ s=1,\ldots,q$ of matrix $A(t)$ there are no integral relations of the form /3/

$$\langle \Lambda P^{(j)}\rangle = \frac{2\pi\sqrt{-1}}{\omega}p_j^*, \quad p_j^* = 0, \pm 1, \pm 2, \ldots$$
$$P^{(j)} = (p_1^{(j)},\ldots,p_q^{(j)}), \quad |P^{(j)}| = p_1^{(j)}+\ldots+p_q^{(j)} = m_j \geqslant 2,\ p_s^{(j)}\geqslant 0, \quad \Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_q), \quad j=1,\ldots,\mu, \quad \mu\geqslant 1 \tag{0.2}$$

where $P^{(j)}$ is a q-dimensional vector with integral components $P_s^{(j)}$. If (0.2) is satisfied, we say that one or several internal resonances are present. In that case the problem reduces to that of stability of the zero solution of a system of autonomous equations with $(r+2q)$ zero roots /2/. Certain problems of stability were considered for that case in /4—7/.

The present investigation is aimed at obtaining the standard form of system (0.1) with (0.2) satisfied up to the m-th order $(m\geqslant 2)$ term in the critical case of r zero and $2q$ pure imaginary characteristic indices with simple elementary divisors and, also, at the investigation of stability of the zero solution of system (0.1) for $r=1$ and $q=1$ when one of relations (0.2) applies (the case when $r=1$ and $q=1$ in systems of general form in the absence of resonances was investigated by Kamenkov /2/, and that of $r=2$ and $q=1$ in Hamiltonian systems was analyzed in /8/).

1. The standard form of equations. We assume that (0.2) is satisfied and μ is an arbitrary finite positive integer (apart from (0.2) there are no internal resonances in the system). As shown in /9/, it is possible by using the nonsingular linear transformation with periodic (of period ω) coefficients, to represent system (0.1), without affecting the stability problem, in the form

$$z^{\cdot} = \Lambda z + \sum_{l\geqslant 2} Z^{(l)}(z,\bar z,y,t), \quad z=(z_1,\ldots,z_q)', \qquad y^{\cdot} = \sum_{l\geqslant 2} Y^{(l)}(z,\bar z,y,t), \quad y=(y_1,\ldots,y_r)' \tag{1.1}$$

where z and $\bar z$ are complex conjugate vectors, and $Z^{(l)}$ and $Y^{(l)}$ are vector functions whose components are homogeneous l-th order forms with periodic (of period ω) coefficients.

We derive the standard form of Eqs. (1.1) /10/ by carrying out the normalization not by the cumbersome classical procedure of substitution of series into series (**) but by the method

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**) Medvedev, S. V., On an algoritm of non-Hamiltonian system normalization. VINITI No. 2719-78, 1978.

of Deprit—Hori /11,12/ which is convenient for computer calculations *). Extension of that method to non-Hamiltonian systems is given in /12,13/. Below, we obtain another extension of this method using the Mersman method /1,14/.

We recall that the normalizing transformation of coordinates is defined by formula

$$Q^* = \exp(D_{W(Q)})Q, \quad D_{W(Q)} = \sum_{\beta=1}^{r+2q+1} W_\beta(Q)\frac{\partial}{\partial Q_\beta} \tag{1.2}$$

$$Q^* = (z_1, \ldots, z_q, \bar z_1, \ldots, \bar z_q, y_1, \ldots, y_r, t), \quad Q = (u_1, \ldots, u_q, v_1, \ldots, v_q, w_1, \ldots, w_r, t)$$

where Q^* and Q are $(r+2q+1)$-dimensional vectors of the old and new variables and $D_{W(Q)}$ is a Lie operator with generator $W(Q)$ /12,13/ (time t is used as supplementary coordinate, and $W_{r+2q+1}(Q) \equiv 0$).

The Lie generator $W(Q)$ is in form of the sum of homogeneous forms with periodic in t coefficients

$$W(Q) = \sum_{i\geqslant 1} W_i(Q), \quad W_{\beta,i}(Q) = \sum_{|K_\beta|+|L_\beta|+|N_\beta|=i} B_\beta^*(t)\, u^{K_\beta} v^{L_\beta} w^{N_\beta}, \quad \beta = 1, \ldots, r+2q \tag{1.3}$$

$$B_\beta^*(t+\omega) = B_\beta^*(t), \quad |K_\beta| = \sum_{s=1}^{q} k_{s\beta}, \quad |L_\beta| = \sum_{s=1}^{q} l_{s\beta}, \quad (*) \Rightarrow (K_\beta, L_\beta, N_\beta)$$

$$|N_\beta| = \sum_{s=1}^{r} n_{s\beta}, \quad u^{K_\beta} v^{L_\beta} w^{N_\beta} = u_1^{k_{1\beta}} \ldots u_q^{k_{q\beta}} v_1^{l_{1\beta}} \ldots v_q^{l_{q\beta}} w_1^{n_{1\beta}} \ldots w_r^{n_{r\beta}}$$

where K_β, L_β, and N_β are vectors with integral components (in what follows we use vector exponents in the presentation of homogeneous forms, similarly to formula (1.3).

It was shown in /12,13/ that the image of any analytic function $g(Q^*)$ after transformation (1.2) is

$$g^*(Q) = \exp(D_{W(Q)})\, g(Q) \tag{1.4}$$

Using the Mersman procedure /1,13/, developed by him for Hamiltonian systems and convenient for problems of normalization of nonlinear ordinary differential equations, it is possible to show that for the determination of the image $g^*(Q)$ in (1.4) it is sufficient use the recurrent relations

$$g^*(Q) = \sum_{n=0}^{\infty} g_n^*(Q) \tag{1.5}$$

$$g_n^*(Q) = \sum_{k=0}^{n} g_{k,n-k}(Q); \; g_{k,n}(Q) = \frac{1}{k}\sum_{m=0}^{n} D_{m+1} g_{k-1,n-m}(Q), \quad g_{0,n}(Q) = g_n(Q); \; k \geqslant 1; \; D_{m+1} = D_{W_{m+1}}(Q) \tag{1.6}$$

where $g_n^*(Q)$ are homogeneous forms of power $(n+1)$ in variables Q with periodic in t coefficients. Formulas (1.6) contain intermediate forms $g_{k-1,n-m}(Q)$.

Substituting in conformity with (1.7) and (1.8) forms $g_{k-1,n-m}$ one into another beginning with $k=1$, it is possible to obtain $g_n^*(Q)$ as a function of the initial forms $g_i(Q)$, $i \in \{0, 1, 2, \ldots\}$ (this is the mentioned above modification of the Deprit—Hori method)

$$g_n^*(Q) = \sum_{k=0}^{n} N_k g_{n-k}(Q), \quad n \geqslant 0, \quad N_k = \sum_{i=1}^{k} \frac{1}{i!} \sum_{|l|=k} \prod_{j=1}^{i} D_{l_j}, \quad N_0 = 1, \quad |l| = \sum_{j=1}^{i} l_j, \quad k \geqslant 1 \tag{1.7}$$

It follows from (1.7) that, if at least one form $g_i(Q) \equiv 0$, $i \in \{1, 2, \ldots\}$, then in the right-hand side of formula for $g_n^*(Q)$ terms $N_k g_i(Q)$ are always absent. By applying (1.5)—(1.7) we reduce the normalization procedure of Eqs. (1.1) to solving the following operator equation:

$$D_i^* g_0(Q) = f_i(Q) - g_i^{**}(Q), \quad i = 1, 2, \ldots, m-1 \tag{1.8}$$

where $g_i(Q)$ and $f_i(Q)$ are vector functions of dimension $(r+2q)$ whose components are forms of the $(i+1)$-st order in the right-hand sides of the input system (1.1) and the normalized system, respectively; forms of the $(i+1)$-st order with known periodic coefficients

$$g_i^{**}(Q) = g_i(Q) + \sum_{k=1}^{i-1}\Big[N_k g_{i-k}(Q) - \sum_{\beta=1}^{r+2q} f_{\beta,i-k}(Q)\frac{\partial}{\partial Q_\beta} N_k Q\Big] +$$
$$\sum_{k=2}^{i}\frac{1}{k!}\Big[\sum_{|l|=i}\prod_{j=1}^{k} D_{l_j} g_0(Q) - \sum_{\beta=1}^{r+2q} f_{\beta,0}(Q)\frac{\partial}{\partial Q_\beta}\sum_{|l|=i}\prod_{j=1}^{k} D_{l_j} Q -$$
$$\frac{\partial}{\partial t}\sum_{|l|=i}\prod_{j=1}^{k} D_{l_j} Q\Big], \quad |l| = \sum_{s=1}^{k} l_s, \; l_j \geqslant 1, \; i > 1, \; g_1^{**}(Q) = g_1(Q)$$

play the part of components of the vector function $g_i^{**}(Q)$ of the same dimension as $g_i(Q)$ and

*) Markeev, A. P. and Sokol'skii, A. G., Certain computational algorithm of Hamiltonian system normalization. Preprint No.31 Inst. Appl. Math. Akad.Nauk SSSR, 1976.

$f_i(Q)$.

The effect of operator D_i^* on first order forms $g_0(Q)$ is

$$D_i^* g_0(Q) = D_i g_0(Q) - \sum_{\beta=1}^{r+2q} g_{\beta,0}(Q) \frac{\partial W_i(Q)}{\partial Q_\beta} - \frac{\partial W_i(Q)}{\partial t}$$

Note that first order terms are not affected by normalization so that

$$g_{\alpha,0}(Q) = \sqrt{-1}\ \omega_\alpha u_\alpha, \quad g_{\alpha+q,0}(Q) = \bar{g}_{\alpha,0}(Q), \quad g_{\gamma,0}(Q) = 0,\ \alpha = 1,\ldots, q,\ \gamma = 1,\ldots, r$$

Taking this into account, we represent the left-hand side of (1.8) thus:

$$D_i^* g_{\alpha,0}(Q) = -W_{\alpha,i}(Q)\langle K_\alpha - L_\alpha - E_\alpha, \lambda\rangle - \frac{\partial W_{\alpha,i}(Q)}{\partial t}, \quad D_i^* g_{\gamma,0}(Q) = -W_{\gamma,i}(Q)\langle K_\gamma - L_\gamma, \lambda\rangle - \frac{\partial W_{\gamma,i}(Q)}{\partial t}$$

$$E_\alpha = (0,\ldots, 1,\ldots, 0), \quad \lambda = (\lambda_1,\ldots, \lambda_q)'$$

(here and subsequently only formulas for the first group of complex conjugate variables $u_1, \ldots, u_q$) are written out). The coefficients $B_\alpha^*(t)$ of forms $W_i(Q)$ which determine the normalizing transformation are, then, determined by equations of the form

$$dB_\alpha^*/dt + \varkappa_\alpha B_\alpha^* = G_\alpha^*(t) - A_\alpha^*(t), \quad \alpha = 1,\ldots, r+2q \tag{1.9}$$

$$\varkappa_\alpha = \langle K_\alpha - L_\alpha - E_\alpha, \lambda\rangle, \quad \alpha = 1,\ldots, q, \quad \varkappa_\alpha = \langle K_\alpha - L_\alpha, \lambda\rangle, \quad \alpha = 1,\ldots, r$$

where $A_\alpha^*(t)$ are the sought coefficients of the standard form, and $G_\alpha^*(t)$ are known periodic functions. The solution of Eqs. (1.9) is known /2/, namely: it follows from (1.9) that when conditions

$$\varkappa_\alpha = \frac{2\pi\sqrt{-1}}{\omega} p_\alpha, \quad p_\alpha = 0, \pm 1, \pm 2, \ldots, \quad \alpha = 1, \ldots, r+2q \tag{1.10}$$

are not satisfied, there exists a unique periodic (of period ω) solution for $B_\alpha^*(t)$

$$B_\alpha^*(t) = \exp(\varkappa_\alpha t)\left[\frac{\exp(\varkappa_\alpha\omega)}{1-\exp(\varkappa_\alpha\omega)}\int_0^\omega \exp(-\varkappa_\alpha t) G_\alpha^*(t)\,dt + \int_0^t \exp(-\varkappa_\alpha\tau) G_\alpha^*(\tau)\,d\tau\right] \tag{1.11}$$

for any $A_\alpha^*(t)$, including $A_\alpha^*(t) = 0$.

The terms of homogeneous forms in the right-hand sides of equations corresponding to such K_α, L_α, and N_α are called nonresonant; they can be suppressed by a suitable selection of coefficients of the normalizing transformation. On the other hand solution (1.11) looses its meaning, if (1.10) holds. But if we set

$$A_\alpha^*(t) = d_\alpha^* \exp(-\varkappa_\alpha t), \quad d_\alpha^* = \frac{1}{\omega}\int_0^\omega \exp(\varkappa_\alpha t) C_\alpha^*(t)\,dt$$

then Eq. (1.9) will again have the unique (but different from (1.11)) periodic solution

$$B_\alpha^*(t) = \exp(-\varkappa_\alpha t)\left[c + \int_0^t [\exp(\varkappa_\alpha\tau) G_\alpha^*(\tau) - d_\alpha^*]\,d\tau\right]$$

where c is an arbitrary constant. The respective terms are called resonant. The successive determination of resonant and nonresonant coefficients $B_\alpha^*(t)$ and $A_\alpha^*(t)$ in conformity with the described above procedure for $i = 2, 3, \ldots, m$ yields each time equations of type (1.9); the right-hand sides of $f_i(Q)$ of the normalized system now contain only resonant terms with constants that are either constant or (in case of degeneration) zero. Coefficients of the normalizing transform (and also of the generator $W(Q)$) will obtain in the form of bounded ω-periodic functions.

Let us determine the structure of resonant terms of standard form. We note, first of all, that Eq. (1.10) admits for any i and $p_\alpha = 0$ the following simple solution:

$$K_\alpha = L_\alpha + E_\alpha, \quad \alpha = 1,\ldots, q; \quad 2\,|L_\alpha| = i - 1 - |N_\alpha| \tag{1.12}$$

$$K_\alpha = L_\alpha, \quad \alpha = 1,\ldots, r; \quad 2\,|L_\alpha| = i - |N_\alpha| \tag{1.13}$$

The respective terms, called terms of identical resonance, are always present in the standard form irrespective of the values of natural frequencies ω_α in any order of i, unlike in the critical case of q pairs of pure imaginary roots, where they appear only in terms of odd order /2/.

If relations (0.2) exist, Eq. (1.10) has in addition to (1.12) and (1.13) two other groups of solutions

$$L_\alpha = \varepsilon P^{(s)} + H_\alpha^{(s)} - E_\alpha,\ K_\alpha = H_\alpha^{(s)},\ p_\alpha = -\varepsilon p_s^*,\ \alpha = 1\ldots, q \tag{1.14}$$

$$L_\beta = \sigma P^{(s)} + H_\beta^{(s)}, \quad K_\beta = H_\beta^{(s)}, \quad p_\beta = -\sigma p_s^*, \quad \beta = 1,\ldots, r$$

$$K_\alpha = \varepsilon P^{(s)} + H_\alpha^{(s)} + E_\alpha,\ L_\alpha = H_\alpha^{(s)},\ p_\alpha = \varepsilon p_s^*,\ \alpha = 1,\dots,q$$
$$K_\beta = \sigma P^{(s)} + H_\beta^{(s)},\ L_\beta = H_\beta^{(s)},\ p_\beta = \sigma p_s^*,\ \beta = 1,\dots,r,\ s = 1,\dots,\mu \tag{1.15}$$

where $H_\alpha^{(s)}$ is the column number α of the s-th integral matrix whose elements are nonnegative and satisfy the conditions

$$2|H_\alpha^{(s)}| = (m \pm 1 - \varepsilon m_s - |N_\alpha|) \geqslant 0,\ \alpha = 1,\dots,\ q,\ \varepsilon = 1, 2,\dots,\ \varepsilon_{1,2}$$
$$2|H_\beta^{(s)}| = (m - \sigma m_s - |N_\beta|) \geqslant 0,\ \beta = 1,\dots,\ r,\ \sigma = 1,\dots,\ \sigma_1,\ s = 1,\dots,\ \mu$$
$$\varepsilon_{1,2} = E\,[(m \pm 1 - |N_\alpha|)/m_s],\ \sigma_1 = E\,[(m - |N_\beta|)/m_s]$$

(subscripts 1 and 2 and the plus and minus signs relate to (1.14) and (1.15), respectively). Formulas (1.14) and (1.15) enable us to determine the most general structure of the standard form of Eqs. (1.1) up to terms of m-th order $(m \geqslant 2)$. Note that terms of the s-th internal resonance of order m_s can appear only in forms of order not lower than $m_s - 1$.

We pass from variables u_α, v_α, and w_α to new variables ρ_α, and φ_α using formulas

$$u_\alpha = \rho_\alpha \exp(\sqrt{-1}\,\varphi_\alpha + \sqrt{-1}\,\omega_\alpha t),\quad v_\alpha = \bar{u}_\alpha,\ w_\beta = w_\beta,\quad \alpha = 1,\dots,q,\ \beta = 1,\dots,r$$

The standard form of Eqs. (1.1) is now

$$\begin{aligned}
\rho_\alpha^{\cdot} &= \rho_\alpha \sum_{l\geqslant 2}^{m} \sum_{2|L_\alpha| = l-1-|N_\alpha|} a_\alpha^* \rho^{2L_\alpha} w^{N_\alpha} + \\
&\sum_{s=1}^{\mu} \sum_{\substack{l=m_s-1\\ l\leqslant m\\ m_s\neq m}}^{m} \sum_{\varepsilon=1}^{\varepsilon_1} \rho^{\varepsilon P^{(s)} - E_\alpha} \sum_{2|H_\alpha^{(s)}| = l+1-\varepsilon m_s - |N_\alpha|} P_\alpha^*(\varepsilon\theta_s)\, \rho^{2H_\alpha^{(s)}} w^{N_\alpha} + \\
&\sum_{s=1}^{\mu} \sum_{\substack{l=m_s+1\\ l\leqslant m\\ m_s\neq m}} \sum_{\varepsilon=1}^{\varepsilon} \rho^{\varepsilon P^{(s)} + E_\alpha} \sum_{2|H_\alpha^{(s)}| = l-1-\varepsilon m_s - |N_\alpha|} P_\alpha^*(\varepsilon\theta_s)\, \rho^{2H_\alpha^{(s)}} w^{N_\alpha} + \dots
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
\theta_s^{\cdot} &= \sum_{\alpha=1}^{q} p_\alpha^{(s)} \Big\{ \sum_{l\geqslant 2}^{m} \sum_{2|L_\alpha| = l-1-|N_\alpha|} b_\alpha^* \rho^{2L_\alpha} w^{N_\alpha} + \\
&\sum_{s=1}^{\mu} \sum_{\substack{l=m_s-1\\ l\leqslant m\\ m_s\neq m}}^{m} \sum_{\varepsilon=1}^{\varepsilon_1} \rho^{\varepsilon P^{(s)} - 2E_\alpha} \sum_{2|H_\alpha^{(s)}| = l+1-\varepsilon m_s - |N_\alpha|} \frac{dP_\alpha^*(\varepsilon\theta_s)}{d(\varepsilon\theta_s)} \rho^{2H_\alpha^{(s)}} w^{N_\alpha} + \\
&\sum_{s=1}^{\mu} \sum_{\substack{l=m_s+1\\ l\leqslant m\\ m_s\neq m}}^{m} \sum_{\varepsilon=1}^{\varepsilon_2} \rho^{\varepsilon P^{(s)}} \sum_{2|H_\alpha^{(s)}| = l-1-\varepsilon m_s - |N_\alpha|} \frac{dP_\alpha^*(\varepsilon\theta_s)}{d(\varepsilon\theta_s)} \rho^{2H_\alpha^{(s)}} w^{N_\alpha} + \dots \Big\}
\end{aligned}$$

$$\begin{aligned}
w_\gamma^{\cdot} &= \sum_{l\geqslant 2}^{m} \sum_{2|L_\gamma| = l-|N_\gamma|} a_\gamma^* \rho^{2L_\gamma} w^{N_\gamma} + \\
&\sum_{s=1}^{\mu} \sum_{l=m_s}^{m} \sum_{\sigma=1}^{\sigma_1} \rho^{\sigma P^{(s)}} \sum_{2|H_\gamma^{(s)}| = l-\sigma m_s - |N_\gamma|} Q_\gamma^*(\sigma\theta_s)\, \rho^{2H_\gamma^{(s)}} w^{N_\gamma} + \dots
\end{aligned}$$

$$\alpha = 1,\dots,q;\quad \gamma = 1,\dots,r;\quad \theta_s = \sum_{\alpha=1}^{q} p_\alpha^{(s)} \varphi_\alpha;\quad s = 1,\dots,\mu$$
$$P_\alpha^*(\varepsilon\theta_s) = a_\alpha^* \cos(\varepsilon\theta_s) + b_\alpha^* \sin(\varepsilon\theta_s),\quad a_\alpha^* = \operatorname{Re} d_\alpha^*;$$
$$b_\alpha^* = \operatorname{Im} d_\alpha^*$$
$$Q_\gamma^*(\sigma\theta_s) = d_\gamma^{**} \exp(-\sqrt{-1}\,\sigma\theta_s) + d_\gamma^{***} \exp(\sqrt{-1}\,\sigma\theta_s)$$
$$(*) \Rightarrow (H_\alpha^{(1)},\ \varepsilon P^{(s)} + H_\alpha^{(s)} - E_\alpha,\ N_\alpha)\,(\varepsilon P^{(s)} + H_\alpha^{(s)} + E_\alpha,\quad H_\alpha^{(s)}, N_\alpha)$$
$$(**) \Rightarrow (H_\gamma^{(s)},\ \sigma P^{(s)} + H_\gamma^{(s)}, N_\gamma),\quad (***) \Rightarrow (\sigma P^{(s)} + H_\gamma^{(s)},\ H_\gamma^{(s)}, N_\gamma)$$

where terms of order higher than m which are holomorphic functions periodic in t. It will be seen from (1.16) that in the presence of (0.2) system (1.1) is reduced by normalization to an autonomous system with $r + 2q$ zero roots to which correspond $r + 2q$ groups of solutions. It is interesting to note that the structure of standard form (1.16) does not vary independent of whether relations (0.2) hold for $p_j^* = 0$ or for $p_j^* \neq 0,\ j \in \{1,\dots,\mu\}$. This property was established for the case of $r = 0$ in /4/.

2. **Lemma on instability.** Let us prove the following lemma which is a modification of Kamenkov's theorem on instability /2/. A similar modification was proved earlier in the particular case of two degrees of freedom(*).

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*) Khazina, G. G. and Khazin, L. G., On the possibility of resonance stabilization of a system of oscillators. Preprint, No.130, Inst. Prikl. Matem., Akad.Nauk SSSR, 1978.

Lemma 2.1. Let by some change of variables the equations of perturbed motion be reduced to the form

$$r^{\cdot} = r^m R^{(0)}(\varphi) + r^m \Xi(\varphi, r, t), \quad \varphi_s^{\cdot} = r^{m-1} F_s^{(0)}(\varphi) + r^{m-1} N_s(\varphi, r, t), \quad s = 1, \ldots, \nu \tag{2.1}$$

where $(\nu + 1)$ is the order of system (2.1), m is the order of the first nonzero terms in the right-hand sides, and $R^{(0)}(\varphi)$ and $\Xi(\varphi, r, t)$ are holomorphic periodic functions of the vector of angular variables $\varphi = (\varphi_1, \ldots, \varphi_\nu)$ and, also, r, t; $F_s^{(0)}(\varphi)$, $N_s(\varphi, r, t)$, $s = 1, \ldots, \nu$ are holomorphic periodic functions for all $\varphi \in [0, 2\pi]$ with the possible exclusion of the finite number of singular points; $N_s(\varphi, 0, t) \equiv \Xi(\varphi, 0, t) \equiv 0$.

Let the system of equations

$$F_s^{(0)}(\varphi) = 0, \quad s = 1, \ldots, \nu \tag{2.2}$$

admit the particular real solution for $\varphi_1, \ldots, \varphi_\nu$

$$\varphi = \varphi^{\circ} = \text{const} \tag{2.3}$$

such that point φ° from (2.3) is not singular for functions $F_s^{(0)}(\varphi)$ and $N_s(\varphi, r, t)$.

Then, if

$$R^{(0)}(\varphi^0) > 0 \tag{2.4}$$

the zero solution of (2.1) is Liapunov unstable.

Proof. We linearize in the neighborhood of the particular solution of Eq. (2.1)

$$r^{\cdot} = r^m R^{(0)}(\varphi^{\circ}) + r^m \Phi(r, \eta, t), \quad \eta^{\cdot} = r^{m-1} C\eta + r^{m-1} H_c(r, \eta, t) \tag{2.5}$$

$$\eta = (\eta_1, \ldots, \eta_\nu), \quad \eta_s = \varphi_s - \varphi_s^0, \quad s = 1, \ldots, \nu, \quad C = \| c_{si} \|$$

where functions $\Phi(r, \eta, t)$ and $H_s(r, \eta, t)$ that are holomorphic in some region $B \in R^{\nu+1}$ of the coordinate origin, satisfy the conditions $\Phi(0, 0, t) \equiv H_s(0, 0, t) \equiv 0$. In what follows we use the method proposed by Kamenkov in the proof of extension of the theorem of Briot and Bouquet /2/.

The system of Eqs. (2.5) can be reduced to equations of the form

$$r\frac{d\eta_s}{dr} = \Big[\sum_{i=1}^{\nu} c_{si}\eta_i + rH_s(r, \eta, t)\Big]\Big[R^{(0)}(\varphi^0) + \Phi(r, \eta, t)\Big]^{-1} \tag{2.6}$$

In region B by virtue of (2.4) and of the holomorphy of $\Phi(r, \eta, t)$ it is possible to represent system (2.6) as follows:

$$r\frac{d\eta_s}{dr} = \sum_{i=1}^{\nu} c_{si}^{*}\eta_i + \Gamma_s(r, \eta, t), \quad s = 1, \ldots, \nu \tag{2.7}$$

where $\Gamma_s(r, \eta, t)$ are holomorphic in B functions with coefficients periodic in t (in (2.7) time t is considered to be a parameter).

According to /2/ there exists one determinate system of functions $\eta_s(r, t)$ that satisfy (2.7) and such that $\eta_s(0, t) \equiv 0$; functions $\eta_s(r, t)$ are holomorphic in B with coefficients that are periodic in t or r

$$\eta_s(r, t) = h_s r + r\Psi_s(r, t), \quad s = 1, \ldots, \nu \tag{2.8}$$

or in r and $r \ln r$

$$\eta_s(r, t) = h_s r + r\Psi_s(r, r\ln r, t), \quad s = 1, \ldots, \nu \tag{2.9}$$

Substitution of solution (2.8) or (2.9) into the first of Eqs. (2.5) yields

$$r^{\cdot} = r^m R^{(0)}(\varphi^0) + r^m H(r, t), \quad H(0, t) \equiv 0 \tag{2.10}$$

where $H(r, t)$ is a function holomorphic in B and periodic in t. It follows from (2.10) that when condition (2.4) is satisfied $r(t)$ increases with increasing t independent of m and the unperturbed motion that corresponds to system (2.1) is unstable.

3. Investigation of stability in the case of resonance with $r = 1$ and $q = 1$.

We assume the existence of internal resonance of the form

$$3\lambda = \frac{2\pi\sqrt{-1}}{\omega}p, \quad p = \pm 1, \pm 2, \ldots \tag{3.1}$$

The case of absence of relation (3.1) was considered in detail in /2/.

In conformity with (1.10) the standard form of equations is

$$\rho^{\cdot} = D\rho^2\cos(\psi - 3\theta) + \alpha\rho w + \ldots, \quad \theta^{\cdot} = D\rho\sin(\psi - 3\theta) + \beta w + \ldots, \quad w^{\cdot} = dw^2 + c\rho^2 + \ldots \tag{3.2}$$

$$D = (a^2 + b^2)^{1/2}, \quad \sin\psi = a / D, \quad \cos\psi = b / D$$

where $a, b, \alpha, \beta, c, d$ are real coefficients determiend by formulas in Sect.1 in terms of coefficients of the input system. It can be shown that when $d \neq 0$ the zero solution is Liapunov unstable, using Kamenkov's theorem on instability /2/ for proving this. Consequently we henceforth assume that $d = 0$. We pass from ρ and w to polar coordinates r and φ. In the new coordinates system (3.2) assumes the form

$$\begin{aligned} r^{\cdot} &= r^2R^{(0)}(\varphi, \theta) + \ldots + r^{m+2}R^{(m)}(\varphi, \theta) + r^{m+2}H(\varphi, \theta, r, t) \\ \varphi^{\cdot} &= rF_1^{(0)}(\varphi, \theta) + \ldots + r^{m+1}F_1^{(m)}(\varphi, \theta) + r^{m+1}\Phi_1(\varphi, \theta, r, t) \\ \theta^{\cdot} &= rF_2^{(0)}(\varphi, \theta) + \ldots + r^{m+1}F_2^{(m)}(\varphi, \theta) + r^{m+1}\Phi_2(\varphi, \theta, r, t) \end{aligned} \tag{3.3}$$

where $R^{(l)}(\varphi, \theta)$, $F_{1,2}^{(l)}(\varphi, \theta)$ $(l = 0, 1, \ldots, m)$, $H(\varphi, \theta, r, t)$, and $\Phi_{1,2}(\varphi, \theta, r, t)$ are holomorphic functions periodic in φ, θ or φ, θ, t, and

$$R^{(0)}(\varphi, \theta) = (c + \alpha)\cos\varphi\sin^2\varphi + D\sin^3\varphi\cos(\psi - 3\theta)$$

$$F_1^{(0)}(\varphi, \theta) = \sin\varphi\,[\alpha\cos^2\varphi - c\sin^2\varphi + D\cos\varphi\sin\varphi\cos(\psi - 3\theta)]$$

$$F_2^{(0)}(\varphi, \theta) = D\sin\varphi\sin(\psi - 3\theta) + \beta\cos\varphi$$

It can be seen that the system of equations

$$F_i^{(0)}(\varphi, \theta) = 0, \quad i = 1, 2 \tag{3.4}$$

has a real solution when condition

$$D^4 + 4c\alpha D^2 - 4c^2\beta^2 \geqslant 0 \tag{3.5}$$

is satisfied.

The solution $\varphi = \varphi^\circ = \text{const}$, $\theta = \theta^\circ = \text{const}$ of (3.4) has the property that when $\varphi^\circ \in [0, \pi/2]$, it has another solution $\varphi^{*\circ} = \varphi^\circ + \pi$ and vice versa. The quantity $R^{(0)}(\varphi^\circ, \theta^0)$ may then be positive. According to Lemma 2.1 we have Liapunov instability.

Remark. The case when condition (3.5) is not satisfied and there is, consequently, no real solution of (3.4) requires separate investigation.

Let us prove the following theorem.

Theorem 3.1. Let in (3.2) $d = 0$ and $\beta = 0$. If the inequality

$$D^2 \geqslant -4c\alpha \tag{3.6}$$

is satisfied, the zero solution of (3.2) is Liapunov unstable. If (3.6) is violated, still for $\varphi = \varphi^\circ$ and $\theta = \theta^\circ$, which are solutions of (3.4), the relation

$$R^0(\varphi^\circ, \theta^\circ) = \ldots = R^{(m-1)}(\varphi^\circ, \theta^\circ) = 0, \quad R^{(m)}(\varphi^\circ, \theta^\circ) > 0, \quad m \geqslant 1 \tag{3.7}$$

is valid, and we have instability. If, however, (3.6) is not satisfied and

$$R^{(0)}(\varphi^\circ, \theta^\circ) = \ldots = R^{(m-1)}(\varphi^\circ, \theta^\circ) = 0, \quad R^{(m)}(\varphi^\circ, \theta^\circ) < 0, \quad m \geqslant 1 \tag{3.8}$$

the zero solution is asymptotically Liapunov stable.

Proof. As the Liapunov function we take

$$V = r\exp\left\{N\left[-\frac{\alpha + c}{3}\cos^3\varphi + c\cos\varphi + \frac{D}{3}\sin^3\varphi\cos(\psi - 3\theta)\right]\right\} \tag{3.9}$$

By virtue of equations of perturbed motion the time derivative $V^{\cdot}$ is

$$V^{\cdot} = Vr\Big\{R^{(0)}(\varphi, \theta) + N[F_1^{(0)}(\varphi, \theta)]^2 + N[F_2^{(0)}(\varphi, \theta)]^2 + \sum_{k=1}^{2} F_k^{(0)}(\varphi, \theta)\Phi_k(\varphi, \theta, r, t) + \sum_{k=1}^{m} r^kR^{(k)}(\varphi, \theta) + r^mH(\varphi, \theta, r, t)\Big\} \tag{3.10}$$

When condition (3.6) or (3.7) is satisfied, $R^{(m)}(\varphi^\circ, \theta^\circ) > 0$ so that function $V^{\cdot}$ can be made positive definite for all φ, $\theta \in [0, 2\pi]$, including those for which $F_{1,2}^{(0)}(\varphi, \theta) = 0$, if N is made a fairly large positive number. Then function V satisfies Liapunov's theorem on instability /9/.

If condition (3.8) is satisfied and $D^2 < -4c\alpha$, it is possible to show that $R^{(0)}(\varphi^\circ, \theta^\circ) = 0$. If we select a fairly large positive number for N, then function $V^{\cdot}$ can be made negative definite throughout the region where $F_{1,2}^{(0)}(\varphi, \theta) \neq 0$. For those φ and θ for which $F_{1,2}^{(0)}(\varphi, \theta) = 0$ we have (3.8) so that the sign of expressions between braces in (3.10) are, as previously, negative. Thus V satisfies Liapunov's theorem on asymptotic stability. The theorem is proved.

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